Appendix: Critical Fracture Stress of a Plate with an Inclined Crack

Consider a large thick plate of steel contains a crack of length $2a = 5\ mm$ oriented at an angle $\beta = 30°$ with respect to the direction of applied uniaxial stress σ (Figure 27).Assuming $K_{Ic} = 60\ MPa\sqrt{m}$, $E = 210\ GPa$, $\nu = 0.3$, here we calculate the angle of initial crack extension and the value of the critical stress σ_c for crack growth according to the strain energy density criterion when the stress σ is (a) tensile and (b) compressive.

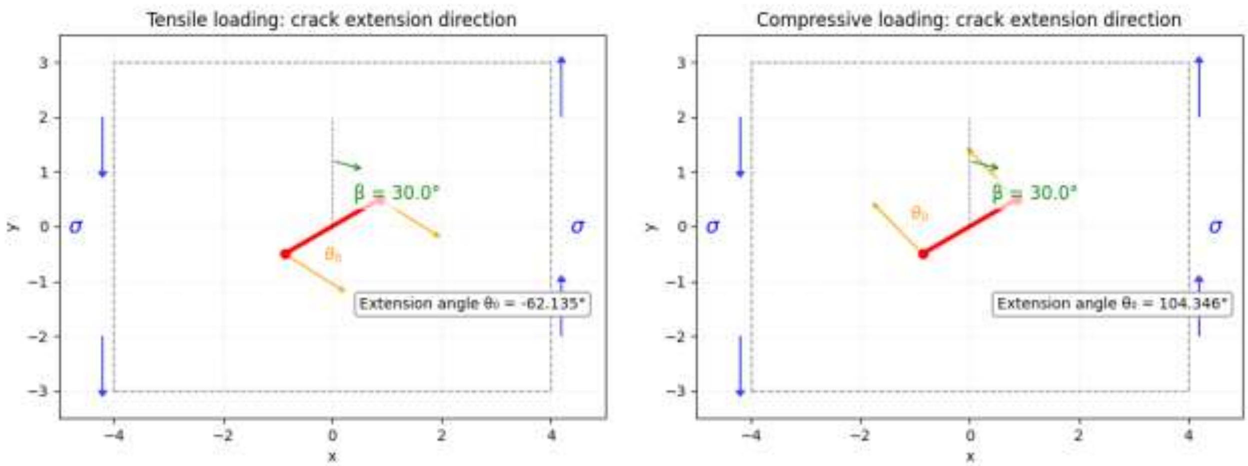

Figure 27. Geometry of an inclined centre crack in an infinite plate under uniaxial tensile stress.

The angle θ_0 of initial crack extension is calculated according to the strain energy density criterion by the relations :

$$\frac{\partial S}{\partial \theta} = 0, \quad \frac{\partial^2 S}{\partial \theta^2} > 0,$$

where S is the strain energy density factor and (r, θ) are polar coordinates measured from the crack tip.

S is given by:

$$S = a_{11}k_I^2 + 2a_{12}k_I k_{II} + a_{22}k_{II}^2,$$

where the coefficients a_{ij} $(i, j = 1,2)$ are:

$$\begin{aligned} 16\mu a_{11} &= (1 + \cos\theta)(\kappa - \cos\theta), \\ 16\mu a_{12} &= \sin\theta[2\cos\theta - (\kappa - 1)], \\ 16\mu a_{22} &= (\kappa + 1)(1 - \cos\theta) + (1 + \cos\theta)(3\cos\theta - 1). \end{aligned}$$

In these equations, μ is the shear modulus, $k_i = K_i/\sqrt{\pi}$ $(i = I, II)$, where K_I and K_{II} are the mode-I and mode-II stress intensity factors, and $\kappa = 3 - 4\nu$ for plane strain, or $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

Crack initiation occurs when:

$$S_{\min} = S_c,$$

where S_c is the critical value of the strain energy density factor, a material constant given by:

$$S_c = \frac{(1+\nu)(1-2\nu)K_{Ic}^2}{2\pi E}.$$

Consider a central crack of length $2a$ in an infinite plate subjected to a uniform uniaxial stress σ at infinity; suppose the axis of the crack makes an angle β with the direction of stress σ. The stress intensity factors are:

$$K_I = \sigma\sqrt{\pi a}\sin^2\beta, \quad K_{II} = \sigma\sqrt{\pi a}\sin\beta\cos\beta.$$

The crack extension angle θ_c is obtained from:

$$(\kappa - 1)\sin(\theta_c - 2\beta) - 2\sin[2(\theta_c - \beta)] - \sin 2\theta_c = 0, \quad \beta \neq 0.$$

For a thick plate, plane strain conditions dominate: $\kappa = 3 - 4\nu = 3 - 4(0.3) = 1.8$. With $\beta = 30°$, Equation (9) becomes:

$$0.8\sin(\theta_c - 60°) - 2\sin[2(\theta_c - 30°)] - \sin 2\theta_c = 0.$$

The first root of this equation corresponding to tensile applied stress σ which satisfies $\partial^2 S/\partial\theta^2 > 0$ is:

$$\boxed{\theta_c = -62.135°}.$$

This root yields:

$$\frac{1}{\sigma^2}\frac{\partial^2 S}{\partial\theta^2} = 4.69 > 0.$$

The critical value S_c is:

$$S_c = \frac{(1+0.3)(1-0.6)(60)^2}{2\pi(210\times 10^3)} = 1.42\times 10^{-3}\ \text{MPa}\cdot\text{m}.$$

The shear modulus μ is:

$$\mu = \frac{E}{2(1+\nu)} = \frac{210}{2(1.3)} = 80.77 \text{ GPa.}$$

The coefficients a_{ij} at $\theta_c = -62.135°$ are:

$$\begin{aligned} 16\mu a_{11} &= 1.51314 \times 10^{-6} \text{ MPa}^{-1}, \\ 16\mu a_{12} &= -0.0922 \times 10^{-6} \text{ MPa}^{-1}, \\ 16\mu a_{22} &= 1.61063 \times 10^{-6} \text{ MPa}^{-1}. \end{aligned}$$

The strain energy density factor is:

$$\begin{aligned} S(\theta_c) = & \frac{\sigma^2(2.5 \times 10^{-3})}{16(80.77 \times 10^3)}[1.51314\sin^2 30° \\ & +2(-0.0922)\sin30°\cos30° + 1.61063\cos^2 30°] \\ & = 0.9415 \times 10^{-9}\sigma^2 \text{ MPa}^{-1}\text{m.} \end{aligned}$$

Setting $S(\theta_c) = S_c$ gives:

$$0.9415 \times 10^{-9}\sigma_c^2 = 1.42 \times 10^{-3},$$

$$\boxed{\sigma_c = 1228 \, MPa}.$$

The initial crack extension angle is $\theta_c = -62.135°$.

The second root of Equation (10) corresponding to compressive stress σ is:

$$\boxed{\theta_c = 104.346°}.$$

The second derivative condition gives:

$$\frac{1}{\sigma^2}\frac{\partial^2 S}{\partial \theta^2} = 5.744 > 0.$$

The coefficients a_{ij} at $\theta_c = 104.346°$ are:

$$\begin{aligned} 16\mu a_{11} &= 1.5404, \\ 16\mu a_{12} &= -1.255, \\ 16\mu a_{22} &= 2.1824. \end{aligned}$$

Then:

$$S(\theta_c) = 0.4522 \times 10^{-9}\sigma_c^2,$$

and setting $S(\theta_c) = S_c$ yields:

$$\boxed{\sigma_c = 1772\ MPa}.$$

The initial crack extension angle is $\theta_c = 104.346°$. The resulting initial crack extension angles are θc $=-62.135°$ under tension and θc=104.346° under compression, as illustrated in Figure 28.

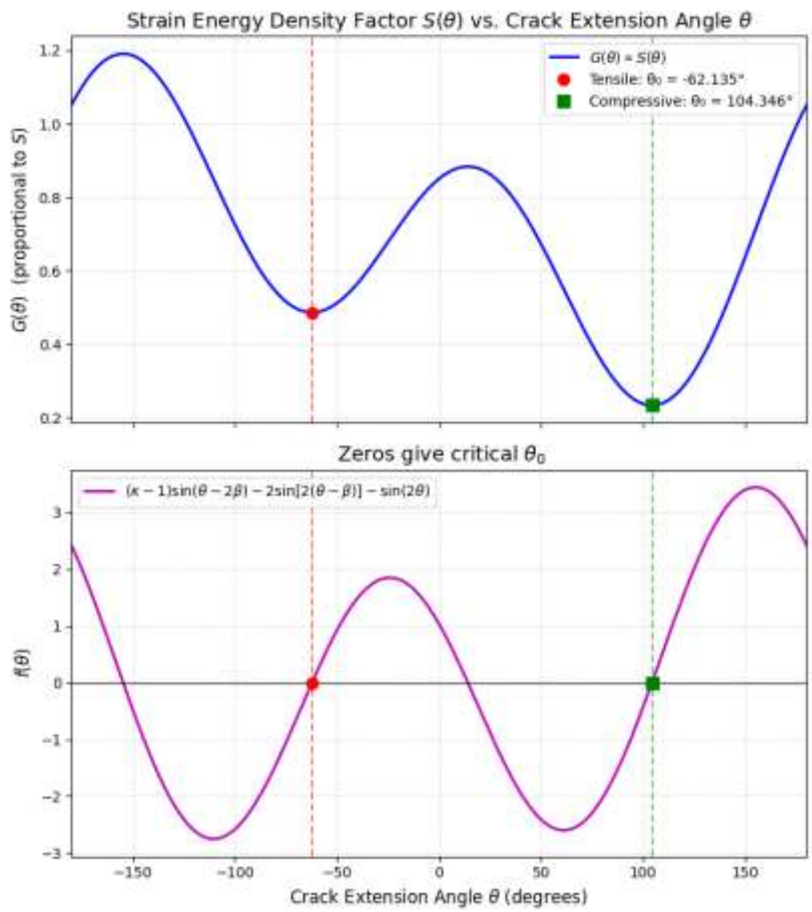

Figure 28. Predicted initial crack extension angles for tensile and compressive loading according to the strain energy density criterion.